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# ASYMPTOTIC BEHAVIOR OF LEAST ENERGY SOLUTIONS TO A FOUR-DIMENSIONAL BIHARMONIC SEMILINEAR PROBLEM

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## Abstract

In this paper, we study the following fourth order elliptic problem  $(E_p)$ :

$$(E_p) \begin{cases} \Delta^2 u = u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0 \end{cases}$$

where  $\Omega$  is a smooth bounded domain in  $\mathbf{R}^4$ ,  $\Delta^2 = \Delta\Delta$  is a biharmonic operator and  $p > 1$  is any positive number.

We investigate the asymptotic behavior as  $p \rightarrow \infty$  of the least energy solutions to  $(E_p)$ . Combining the arguments of Ren-Wei [8] and Wei [10], we show that the least energy solutions remain bounded uniformly in  $p$ , and on convex bounded domains, they have one or two “peaks” away from the boundary. If it happens that the only one peak point appears, we further prove that the peak point must be a critical point of the Robin function of  $\Delta^2$  under the Navier boundary condition.

## 1. Introduction.

In this paper, we study the following fourth order elliptic problem  $(E_p)$ :

$$(E_p) \begin{cases} \Delta^2 u = u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0 \end{cases}$$

where  $\Omega$  is a smooth bounded domain in  $\mathbf{R}^4$ ,  $\Delta^2 = \Delta\Delta$  is a biharmonic operator and  $p > 1$  is any positive number. Boundary condition imposed in  $(E_p)$  is sometimes called the Navier boundary condition.

One of motivation to study such a problem involving biharmonic operator in  $\mathbf{R}^4$  comes from the recent development in conformal geometry on four-manifold. See [3].

On the other hand, problem  $(E_p)$  may be regarded as a natural extension to a higher dimensional case, of the two-dimensional problem treated by Ren and Wei [8], [9].

Ren and Wei considered the least energy solution  $u_p$  of the following semilinear

problem

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases}$$

where  $\Omega$  is a bounded smooth domain  $\subset \mathbf{R}^2$ . They studied the asymptotic behavior of  $u_p$  as the nonlinear exponent  $p \rightarrow \infty$ , and they proved that the least energy solutions remain bounded in  $L^\infty$ -norm regardless of  $p$ . On the shape of solutions, they showed that the least energy solutions must develop one “peak” in the interior of  $\Omega$ , that is the graph of  $u_p$  is becoming like a single spike as  $p \rightarrow \infty$ . Moreover they showed that this peak point must be a critical point of the Robin function of the domain.

Now, in this paper we investigate that whether the analogues of the results of Ren and Wei would hold to the higher dimensional fourth order problem  $(E_p)$ .

Since the complete structure of solution set of the simple-looking problem  $(E_p)$  is widely open, so we restrict our attention, as Ren and Wei did, to the least energy solution constructed as follows.

Let us consider the constrained minimization problem:

$$(1.1) \quad C_p^2 := \inf \left\{ \int_{\Omega} |\Delta u|^2 dx : u \in H^2 \cap H_0^1(\Omega), \|u\|_{p+1} = 1 \right\}.$$

Since the Sobolev imbedding  $H^2 \cap H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$  is compact for any  $p > 1$ , we have at least one minimizer  $\underline{u}_p$  for the problem (1.1), where  $\underline{u}_p \in H^2 \cap H_0^1(\Omega)$ ,  $\|\underline{u}_p\|_{p+1} = 1$ .

Without losing generality, we may assume  $\underline{u}_p > 0$ . Indeed, let  $v$  solves

$$\begin{cases} -\Delta v = |\Delta \underline{u}_p| & \text{in } \Omega, \\ v|_{\partial\Omega} = 0. \end{cases}$$

Then the maximum principle implies  $v \geq |\underline{u}_p|$ , so we have

$$C_p^2 = \frac{\int_{\Omega} |\Delta \underline{u}_p|^2 dx}{\left(\int_{\Omega} |\underline{u}_p|^{p+1} dx\right)^{2/(p+1)}} \geq \frac{\int_{\Omega} |\Delta v|^2 dx}{\left(\int_{\Omega} v^{p+1} dx\right)^{2/(p+1)}},$$

that is, the positive function  $v$  also minimizes  $C_p^2$ .

Set

$$(1.2) \quad u_p := C_p^{2/(p-1)} \underline{u}_p,$$

then  $u_p$  solves  $(E_p)$  and  $C_p = \|\Delta u_p\|_{L^2} / \|u_p\|_{L^{p+1}}$ . Standard regularity argument implies that any weak solution  $u \in H^2 \cap H_0^1(\Omega)$  satisfies  $\Delta u \in W^{2,\sigma} \cap W_0^{1,\sigma}(\Omega)$  ( $\forall \sigma > 1$ ) (for example, see [4]). Therefore  $u_p$  is smooth and  $u_p = \Delta u_p = 0$  on  $\partial\Omega$ .

We call  $u_p$  the *least energy solution* to  $(E_p)$ .

Our first result is the same as the one in Ren and Wei.

**Theorem 1.1.** *Let  $u_p$  be a least energy solution to  $(E_p)$ . Then there exist  $C_1, C_2$  (independent of  $p$ ), such that*

$$0 < C_1 < \|u_p\|_{L^\infty} < C_2 < \infty$$

for  $p$  large enough.

To state further results, we need some definitions. Set

$$(1.3) \quad w_p := \frac{u_p}{\int_{\Omega} u_p^p dx}.$$

For a sequence  $w_{p_n}$  of  $w_p$ , we define the *blow up set*  $S$  of  $\{w_{p_n}\}$  as usual:

$$S := \{x \in \overline{\Omega} : \exists \text{ a subsequence } w_{p'_n}, \exists \{x_n\} \subset \Omega \text{ such that} \\ x_n \rightarrow x \text{ and } w_{p'_n}(x_n) \rightarrow \infty\}.$$

We also define a *peak point*  $P$  for  $u_p$  to be a point in  $\overline{\Omega}$  such that  $u_p$  does not vanish in the  $L^\infty$  norm in any small neighborhood of  $P$  as  $p \rightarrow \infty$ . It turns out later that the set of peak points of  $\{u_p\}$  is contained in the blow up set of  $\{w_p\}$ .

**Theorem 1.2.** *Let  $\Omega \subset \mathbf{R}^4$  be a smooth convex bounded domain. Then for any sequence  $w_{p_n}$  of  $w_p$  with  $p_n \rightarrow \infty$ , there exists a subsequence (still denoted by  $w_{p_n}$ ) such that the blow up set  $S$  of this subsequence is contained in  $\Omega$  and has the property  $1 \leq \text{card}(S) \leq 2$ .*

*If  $\text{card}(S) = 1$  and  $S = \{x_0\}$  (one point blow up), then:*

$$(1) \quad f_n := \frac{u_{p_n}^{p_n}}{\int_{\Omega} u_{p_n}^{p_n} dx} = \left( \int_{\Omega} u_{p_n}^{p_n} dx \right)^{p_n-1} w_{p_n}^{p_n} \xrightarrow{*} \delta_{x_0}$$

*in the sense of Radon measures of  $\Omega$ .*

(2)  $w_{p_n} \rightarrow G_4(\cdot, x_0)$  in  $C_{loc}^4(\overline{\Omega} \setminus \{x_0\})$  where  $G_4(x, y)$  denotes the Green function of  $\Delta^2$  under the Navier boundary condition:

$$\begin{cases} \Delta_x^2 G_4(x, y) = \delta_y(x), & x \in \Omega, \\ G_4(x, y)|_{x \in \partial\Omega} = \Delta G_4(x, y)|_{x \in \partial\Omega} = 0. \end{cases}$$

(3)  $x_0$  is a critical point of the Robin function  $R_4(x) = H_4(x, x)$ , where

$$H_4(x, y) := G_4(x, y) + \frac{1}{4\sigma_4} \log |x - y|$$

denotes the regular part of  $G_4$  and  $\sigma_4 = 2\pi^2$  is the volume of the unit sphere  $S^3$  in  $\mathbf{R}^4$ .

REMARK 1.3. (1) In Theorem 1.2, the convexity assumption of  $\Omega$  is needed to derive a uniform boundary estimate of solutions to  $(E_p)$  by the use of Method of Moving Planes (MMP).

In  $\Delta$  case, MMP still works well for non-convex domains through the application of Kelvin transformations, and leads to a uniform boundary estimate. However in our  $\Delta^2$  case, Kelvin transformation does not work well simply because Laplacian of a Kelvin transformed function is not 0 on the boundary. It is unclear that actually the boundary blow-up can occur if  $\Omega$  is not convex.

(2) We conjecture that  $\text{card}(S) = 1$  at least for any convex domain. If the following upper bound

$$M := \limsup_{p \rightarrow \infty} \|u_p\|_{L^\infty(\Omega)} < e,$$

holds true, then we can prove the conjecture affirmatively.

Further we believe that in this case the blow up point  $x_0$  is not only a critical point, but the maximum point of Robin function. In 2-dimensional  $\Delta$  case, this is already proved by Flucher and Wei [5].

## 2. Estimates for $C_p$

First we recall D. Adams' version of higher order Trudinger-Moser inequality [1]. This is a key ingredient in our analysis.

**Theorem ([1]).** *Let  $\Omega \subset \mathbf{R}^{2n}$  be a bounded domain and define  $X := H^n(\Omega) \cap \{u : (-\Delta)^j u \in H_0^1(\Omega), j = 0, 1, \dots, [(n-1)/2]\}$ . Then there exists  $C_0 = C_0(n)$  such that*

$$\int_{\Omega} \exp \left( \alpha_{2n} \frac{u^2(x)}{\|(-\Delta)^{n/2} u\|_{L^2(\Omega)}^2} \right) dx \leq C_0 |\Omega|, \quad \forall u \in X.$$

Here,  $\alpha_{2n} = 2^{2n-1} n! (n-1)! \sigma_{2n}$ ,  $\sigma_{2n}$  is the volume of  $S^{2n-1}$  and  $|\Omega|$  is the Lebesgue measure of  $\Omega$ .

In  $\mathbf{R}^4$  ( $n = 2$ ), we have

$$\int_{\Omega} \exp \left( \alpha_4 \frac{u^2(x)}{\|\Delta u\|_{L^2(\Omega)}^2} \right) dx \leq C_0 |\Omega|, \quad \forall u \in H^2 \cap H_0^1(\Omega)$$

where  $\alpha_4 = 16\sigma_4 = 32\pi^2$ .

Using this, we derive the following refined Sobolev imbedding. Though the proof is the same as in [8], we state here for reader's convenience.

**Lemma 2.1** (refinement of the Sobolev imbedding). *For any  $t \geq 2$ , there exists  $D_t > 0$  such that for any  $u \in H^2 \cap H_0^1(\Omega)$ ,*

$$\|u\|_{L^t(\Omega)} \leq D_t t^{1/2} \|\Delta u\|_{L^2(\Omega)}$$

*holds true. Furthermore, we have*

$$\lim_{t \rightarrow \infty} D_t = (32\sigma_4 e)^{-1/2}.$$

*Proof.* Let  $u \in H^2 \cap H_0^1(\Omega)$ . Elementary inequality says that  $x^s/\Gamma(s+1) \leq e^x$  for  $\forall x \geq 0$ ,  $\forall s \geq 0$ , where  $\Gamma(s)$  is the  $\Gamma$  function.

By Adams' version Trudinger-Moser inequality, we have

$$\begin{aligned} & \frac{1}{\Gamma((t/2)+1)} \int_{\Omega} |u|^t dx \\ &= \frac{1}{\Gamma((t/2)+1)} \int_{\Omega} \left( \alpha_4 \frac{u^2}{\|\Delta u\|_{L^2(\Omega)}^2} \right)^{t/2} dx \times \alpha_4^{-t/2} \|\Delta u\|_{L^2(\Omega)}^t \\ &\leq \int_{\Omega} \exp \left( \alpha_4 \frac{u^2(x)}{\|\Delta u\|_{L^2(\Omega)}^2} \right) dx \times \alpha_4^{-t/2} \|\Delta u\|_{L^2(\Omega)}^t \\ &\leq C_0 |\Omega| \alpha_4^{-t/2} \|\Delta u\|_{L^2(\Omega)}^t. \end{aligned}$$

Set

$$D_t := \left( \Gamma \left( \frac{t}{2} + 1 \right) \right)^{1/t} C_0^{1/t} |\Omega|^{1/t} \alpha_4^{-1/2} t^{-1/2}.$$

Then we have

$$\|u\|_{L^t(\Omega)} \leq D_t t^{1/2} \|\Delta u\|_{L^2(\Omega)}.$$

Stirling's formula says that  $(\Gamma((t/2)+1))^{1/t} \sim (t/(2e))^{1/2}$  as  $t \rightarrow \infty$ . So we have

$$\lim_{t \rightarrow \infty} D_t = \left( \frac{1}{2\alpha_4 e} \right)^{1/2},$$

which is a desired result. □

Recall  $C_p$  defined in (1.1). Using the above Lemma and energy comparison, we get the following.

**Lemma 2.2** (asymptotics for  $C_p$ ). *We have*

$$\lim_{p \rightarrow \infty} p^{1/2} C_p = (32\sigma_4 e)^{1/2}.$$

Proof. Lower bound  $(32\sigma_4 e)^{1/2} \leq \liminf_{p \rightarrow \infty} p^{1/2} C_p$  is a direct consequence of Lemma 2.1 and the fact  $C_p = \|\Delta u_p\|_{L^2} / \|u_p\|_{L^{p+1}}$  for least energy solutions  $u_p$ .

Therefore we must prove only the upper bound:  $\limsup_{p \rightarrow \infty} p^{1/2} C_p \leq (32\sigma_4 e)^{1/2}$ . We will do this by constructing a suitable test function for the value  $C_p$ .

We may assume  $0 \in \Omega$  and  $B_L(0) \subset \Omega$ . For  $0 < l < L$ , consider Moser function

$$m_l(|x|) := \begin{cases} \frac{A}{\sigma}, & 0 \leq |x| \leq l, \\ \left(\frac{1}{\sigma A}\right) \log \frac{L}{|x|}, & l \leq |x| \leq L, \\ 0 & \Omega \cap \{L \leq |x|\} \end{cases}$$

where  $A^2 = \log(L/l)$  and  $\sigma = \sqrt{4\sigma_4} = 2\sqrt{2}\pi$ .

Note that the Moser function  $m_l \in H_0^1(\Omega)$  but not in  $H^2(\Omega)$ . So we cannot test the value  $C_p$  by  $m_l$  directly.

Define

$$g_l(|x|) := \begin{cases} 0, & 0 \leq |x| \leq l, \\ \left(\frac{1}{\sigma A}\right) \frac{-2}{|x|^2}, & l \leq |x| \leq L, \\ 0, & \Omega \cap \{L \leq |x|\}. \end{cases}$$

Then  $g_l \in L^2(\Omega)$  and  $\|g_l\|_{L^2(\Omega)} = 1$  by our choice of  $\sigma$ .

Take the unique solution  $a \in H_0^1(\Omega)$ ,  $\tilde{a} \in H_0^1(B_L)$  to the problem

$$\begin{cases} -\Delta a = |g_l| & \text{in } \Omega, \\ a|_{\partial\Omega} = 0 \end{cases}$$

and

$$\begin{cases} -\Delta \tilde{a} = |g_l| & \text{in } B_L, \\ \tilde{a}|_{\partial B_L} = 0 \end{cases}$$

respectively.

By elliptic regularity, we have  $a \in H^2 \cap H_0^1(\Omega)$ . On the other hand, if extended by 0,  $\tilde{a}$  can be regarded as in  $H_0^1(\Omega)$ , but not in  $H^2 \cap H_0^1(\Omega)$ . Also note that  $\tilde{a}$  is a radial function which is explicitly calculate by the formula

$$\tilde{a}(r) = \tilde{a}(0) + \int_{t=0}^{t=r} t^{-3} \left( \int_{s=0}^{s=t} -s^3 |g_l(s)| ds \right) dt$$

for  $0 \leq r = |x| \leq L$ . Indeed, by definition of  $g_l$ , we calculate

$$\tilde{a}(r) = \tilde{a}(0), \quad (0 \leq r \leq l)$$

$$(2.1) \quad \tilde{a}(r) = \tilde{a}(0) + \left( \frac{1}{\sigma A} \right) \left( \frac{1}{2} - \frac{1}{2} \left( \frac{l^2}{r^2} \right) + \log \left( \frac{l}{r} \right) \right). \quad (l \leq r \leq L)$$

Here,  $\tilde{a}(0)$  is determined by the boundary condition  $\tilde{a}(L) = 0$ :

$$(2.2) \quad \tilde{a}(0) = \left( \frac{1}{\sigma A} \right) \left( \frac{1}{2} \left( \frac{l^2}{L^2} \right) - \frac{1}{2} + \log \left( \frac{L}{l} \right) \right).$$

Maximum principle implies that  $a \in H^2 \cap H_0^1(\Omega)$  satisfies  $a > \tilde{a}$  in  $\Omega$ . Therefore  $\|a\|_{L^{p+1}(\Omega)} \geq \|\tilde{a}\|_{L^{p+1}(\Omega)} \geq \|\tilde{a}\|_{L^{p+1}(B_l(0))}$ . Also  $\|\Delta a\|_{L^2(\Omega)} = \|g_l\|_{L^2(\Omega)} = 1$ .

Now, we test the value  $C_p$  with  $a \in H^2 \cap H_0^1(\Omega)$ . Then

$$C_p \leq \frac{\|\Delta a\|_{L^2(\Omega)}}{\|a\|_{L^{p+1}(\Omega)}} \leq \frac{1}{\|\tilde{a}\|_{L^{p+1}(B_l(0))}} = |\tilde{a}(0)|^{-1} |B_l(0)|^{-1/(p+1)}.$$

Setting  $l = L \exp(-(p+1)/8)$ , multiplying  $p^{1/2}$  and letting  $p \rightarrow \infty$  in the above inequality, we have, by using (2.2),  $\limsup_{p \rightarrow \infty} p^{1/2} C_p \leq (32\sigma_4 e)^{1/2}$ . This proves Lemma.  $\square$

Since

$$C_p = \frac{\|\Delta u_p\|_{L^2}}{\|u_p\|_{L^{p+1}}}$$

and

$$\int_{\Omega} |\Delta u_p|^2 dx = \int_{\Omega} u_p^{p+1} dx,$$

we have

$$C_p^2 = \|u_p\|_{p+1}^{p-1}, \quad p \|u_p\|_{p+1}^{p+1} = p C_p^2 C_p^{4/(p-1)}.$$

Therefore Lemma 2.2 implies

**Corollary 2.3.**

$$\lim_{p \rightarrow \infty} p \int_{\Omega} u_p^{p+1} dx = 32\sigma_4 e, \quad \lim_{p \rightarrow \infty} p \int_{\Omega} |\Delta u_p|^2 dx = 32\sigma_4 e.$$

### 3. Proof of Theorem 1.1

First, we show that a uniform lower bound of  $L^\infty$  norm exists for any solution  $u$  to  $(E_p)$ .

Indeed, let  $\lambda_1 > 0$  be the first eigenvalue of  $-\Delta$  acting on  $H_0^1(\Omega)$  and  $\phi_1 > 0$  be a first eigenfunction corresponding to  $\lambda_1$ .

Then, Green's formula implies

$$0 = \int_{\Omega} (u \Delta^2 \phi_1 - \phi_1 \Delta^2 u) dx = \int_{\Omega} (u \lambda_1^2 \phi_1 - \phi_1 u^p) dx,$$

that is,

$$\int_{\Omega} u \phi_1 (\lambda_1^2 - u^{p-1}) dx = 0.$$

Hence  $\lambda_1^2 - u^{p-1}$  must change the sign on  $\Omega$ , which leads to

$$\|u\|_{L^\infty(\Omega)}^{p-1} \geq \lambda_1^2.$$

To obtain a uniform upper bound of  $\|u_p\|_{L^\infty(\Omega)}$ , we use an argument with the coarea formula and the isoperimetric inequality in  $\mathbf{R}^4$ .

Set

$$\gamma_p := \max_{x \in \Omega} u_p(x), \quad \Omega_t := \{x \in \Omega : t < u_p(x)\}, \quad \mathcal{A} := \left\{x \in \Omega : \frac{\gamma_p}{2} < u_p(x)\right\}.$$

By Lemma 2.1 and Corollary 2.3, we have

$$\left(\int_{\Omega} u_p^{4p} dx\right)^{1/(4p)} \leq D_{4p}(4p)^{1/2} \|\Delta u_p\|_{L^2(\Omega)} \leq M$$

where  $M$  is independent of  $p$  if  $p$  large. From this and the Chebyshev inequality we have

$$(3.1) \quad \left(\frac{\gamma_p}{2}\right)^{4p} |\mathcal{A}| \leq M^{4p}.$$

On the other hand, denote  $v_p = -\Delta u_p$ . Integration by parts leads to

$$\int_{\partial\Omega_t} |\nabla u_p| d\sigma = \int_{\Omega_t} v_p dx.$$

Coarea formula says

$$-\frac{d}{dt} |\Omega_t| = \int_{\partial\Omega_t} \frac{d\sigma}{|\nabla u_p|}.$$

Then Schwartz inequality implies

$$-\frac{d}{dt} |\Omega_t| \int_{\Omega_t} v_p dx = \int_{\partial\Omega_t} |\nabla u_p| d\sigma \int_{\partial\Omega_t} \frac{d\sigma}{|\nabla u_p|} \geq |\partial\Omega_t|^2.$$

Note that the isoperimetric inequality in  $\mathbf{R}^4$

$$|\partial E| \geq 4\omega_4^{1/4} |E|^{3/4}$$



where  $E \subset \mathbf{R}^4$  is a Caccioppoli set and  $\omega_4 = |B^4| = |\mathbb{S}^3|/4 = \pi^2/2$  is four dimensional volume of the unit ball. Applying  $E = \Omega_t$ , we have

$$-\frac{d}{dt}|\Omega_t| \int_{\Omega_t} v_p dx \geq 16\omega_4^{1/2}|\Omega_t|^{3/2}.$$

Define  $r(t)$  such that  $|\Omega_t| = \omega_4 r^4(t)$ . Then

$$\frac{d}{dt}|\Omega_t| = 4\omega_4 r^3(t)r'(t).$$

Note that  $r'(t) < 0$ . Hence we have:

$$\begin{aligned} \frac{1}{4\omega_4 r^3(t)} \int_{\Omega_t} v_p dx &\geq -\frac{1}{r'(t)}, \\ -\frac{dt}{dr} &\leq \frac{1}{4\omega_4 r^3(t)} \int_{\Omega_t} v_p dx \leq \frac{1}{4\omega_4 r^3(t)} \left( \sup_{\Omega} v_p \right) |\Omega_t| = \left( \sup_{\Omega} v_p \right) \frac{r}{4}. \end{aligned}$$

Integrating the last inequality from  $r = 0$  to  $r = r_0$ , we have

$$t(0) - t(r_0) \leq \frac{1}{8} \left( \sup_{\Omega} v_p \right) r_0^2.$$

Choose  $r_0$  such that  $t(r_0) = \gamma_p/2$ , that is,  $|\mathcal{A}| = |\Omega_{\gamma_p/2}| = \omega_4 r_0^4$ . Then the above inequality implies

$$\gamma_p \leq \frac{1}{4} \left( \sup_{\Omega} v_p \right) r_0^2.$$

As  $v_p = -\Delta u_p$  satisfies

$$\begin{cases} -\Delta v_p = u_p^p & \text{in } \Omega, \\ v_p|_{\partial\Omega} = 0, \end{cases}$$

we know, by elliptic estimate

$$\sup_{\Omega} v_p \leq C \sup_{\Omega} u_p^p \leq C \gamma_p^p$$

where  $C = C(\Omega)$  (see [6] Theorem 3.7). Thus we have

$$(3.2) \quad \gamma_p \leq \frac{C}{4} \gamma_p^p r_0^2 = \frac{C}{4} \gamma_p^p \left( \frac{1}{\omega_4} |\mathcal{A}| \right)^{1/2}.$$

By (3.1) and (3.2), we have

$$\gamma_p^2 \leq \left( \frac{C}{4} \right)^2 \left( \frac{1}{\omega_4} \right) \gamma_p^{2p} \left( \frac{2M}{\gamma_p} \right)^{4p}$$

which implies

$$\gamma_p \leq \left( \frac{C^2}{16\omega_4} \right)^{1/(2p+2)} (2M)^{2p/(p+1)}.$$

Therefore we conclude that there exists  $C > 0$  (independent of  $p$ ) such that  $\gamma_p \leq C$  for  $p$  large.  $\square$

From Theorem 1.1, we have the following consequence.

**Corollary 3.1.** *There exist  $C_1, C_2 > 0$  independent of  $p$  large such that*

$$\frac{C_1}{p} \leq \int_{\Omega} u_p^p dx \leq \frac{C_2}{p}$$

*holds true.*

*Proof.* By Corollary 2.3 and Theorem 1.1, we have

$$C \leq p \int_{\Omega} u_p^{p+1} dx \leq \|u_p\|_{L^\infty(\Omega)} \times p \int_{\Omega} u_p^p dx \leq C' p \int_{\Omega} u_p^p dx$$

where  $C, C'$  is a positive constant independent of  $p$ .

On the other hand, Hölder inequality implies

$$p \int_{\Omega} u_p^p dx \leq \left( p \int_{\Omega} u_p^{p+1} dx \right)^{p/(p+1)} \times p^{1/(p+1)} |\Omega|^{1/(p+1)}.$$

Note that as  $p \rightarrow \infty$ , RHS of the above inequality is bounded from Corollary 2.3. This proves the conclusion.  $\square$

#### 4. Proof of Theorem 1.2

Set

$$w_p := \frac{u_p}{\int_{\Omega} u_p^p dx} = \frac{u_p}{\lambda_p}, \quad \lambda_p := \int_{\Omega} u_p^p dx,$$

and

$$f_p(x) := \frac{u_p^p(x)}{\int_{\Omega} u_p^p dx}.$$

By our assumption of the convexity of  $\Omega$  and the Method of Moving Plane, it is standard by now to derive the uniform boundary estimate of  $\{w_p\}$  which leads to the fact that the blow up set of  $\{w_p\}$  is contained in the interior of  $\Omega$ ; so we omit the proof of this fact.

At this stage, we recall a useful lemma proved by Wei ([10] Lemma 2.3).

**Lemma 4.1** (Brezis-Merle type  $L^1$  estimate for  $\Delta^2$ ). *Let  $u$  be a  $C^4$  solution of*

$$\begin{cases} \Delta^2 u = f(x) & \text{in } \Omega \subset \mathbf{R}^4, \\ u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0, \end{cases}$$

where  $f \in L^1(\Omega)$ ,  $f \geq 0$ . Then for any  $\varepsilon \in (0, 16\sigma_4)$ , there holds

$$\int_{\Omega} \exp\left(\frac{(16\sigma_4 - \varepsilon)|u(x)|}{\|f\|_{L^1}}\right) dx \leq \frac{16\sigma_4^2}{\varepsilon} |\Omega|.$$

Later we need a variant of ([10] Corollary 2.4) with no boundary condition.

**Corollary 4.2.** *Let  $u_n$  be a sequence of  $C^4$  solutions of  $\Delta^2 u = V_n e^u$  in  $\Omega \subset \mathbf{R}^4$  with no boundary condition. Assume for some  $p \in (1, \infty)$ ,*

- (1)  $\|V_n\|_{L^p(\Omega)} \leq C_1$ ,
- (2)  $\|u_n^+\|_{L^1(\Omega)} \leq C_2$  where  $u_n^+ = \max\{u, 0\}$ ,
- (3)  $\int_{\Omega} |V_n| e^{u_n} dx \leq \exists \varepsilon_0 < 16\sigma_4/p'$

where  $p'$  is a Hölder conjugate exponent of  $p$ . Then,  $\{u_n^+\}$  is uniformly bounded in  $L_{loc}^\infty(\Omega)$ .

The proof of Corollary 4.2 is done along the line of the proof of Corollary 4 in [2], so we omit it here.

From now on, let  $u_n$ ,  $w_n$ ,  $\lambda_n$ ,  $f_n$  denote  $u_{p_n}$ ,  $w_{p_n}$ ,  $\lambda_{p_n}$ ,  $f_{p_n}$  respectively. First, we remark that the blow up set  $S$  of the sequence  $\{w_n\}$  satisfies  $S \neq \emptyset$ . Indeed,

$$\sup_{x \in \bar{\Omega}} w_n(x) \geq \frac{C}{\lambda_n} \rightarrow \infty$$

by Theorem 1.1 and the fact that  $\lambda_n = O(1/p_n) \rightarrow 0$  as  $n \rightarrow \infty$  by Corollary 3.1.

This also shows that the set of peak points of  $\{u_n\}$  is contained in the blow up set of  $\{w_n\}$ .

Note that

$$f_n(x) = \frac{u_n^{p_n}(x)}{\int_{\Omega} u_n^{p_n} dx} \in L^1(\Omega), \quad f_n \geq 0, \quad \int_{\Omega} f_n dx = 1.$$

Then there exists a subsequence (still denoted by  $u_n$ ) such that

$$f_n \xrightarrow{*} \mu, \quad \mu(\Omega) \leq 1$$

in the sense of Radon measures of  $\Omega$ .

Now we define an important quantity

$$(4.1) \quad L_0 := \frac{\limsup_{p \rightarrow \infty} (p \int_{\Omega} u_p^p dx)}{e}.$$

From a little bit more precise estimate in the proof of Theorem 1.1 and Hölder inequality, we know that  $L_0$  satisfies  $1 < L_0 \leq 32\sigma_4$ .

For any  $\delta > 0$ , we call a point  $x_0 \in \Omega$  a  $\delta$ -regular point of  $\{u_n\}$  if there exists  $\varphi \in C_0(\Omega)$ ,  $0 \leq \varphi \leq 1$  with  $\varphi \equiv 1$  near  $x_0$  such that

$$\int_{\Omega} \varphi d\mu < \frac{16\sigma_4}{L_0 + 2\delta}.$$

Further we define for  $\delta > 0$ ,  $\delta$ -irregular set of a sequence  $\{u_n\}$  as

$$\Sigma(\delta) := \{x_0 \in \Omega : x_0 \text{ is not a } \delta\text{-regular point}\}.$$

Note that

$$x_0 \in \Sigma(\delta) \Rightarrow \mu(x_0) \geq \frac{16\sigma_4}{L_0 + 2\delta}.$$

The next result is a key lemma in the proof of Theorem 1.2.

**Lemma 4.3** (smallness of  $\mu$  implies boundedness). *Let  $x_0$  be a  $\delta$ -regular point of a sequence  $\{u_n\}$ , then  $\{w_n\}$  is bounded in  $L^\infty(B_{R_0}(x_0))$  for some  $R_0 > 0$ .*

*Proof.* Let  $x_0$  be a  $\delta$ -regular point. Then there exists  $R > 0$  such that

$$\int_{B_R(x_0)} f_n dx < \frac{16\sigma_4}{L_0 + \delta}$$

for  $n$  sufficiently large.

Split  $w_n = w_{1n} + w_{2n}$ , where  $w_{1n}$  is a solution of

$$\begin{cases} \Delta^2 w_{1n} = f_n & \text{in } B_R(x_0), \\ w_{1n}|_{\partial B_R(x_0)} = \Delta w_{1n}|_{\partial B_R(x_0)} = 0, \end{cases}$$

and  $w_{2n}$  is a solution of

$$\begin{cases} \Delta^2 w_{2n} = 0 & \text{in } B_R(x_0), \\ w_{2n}|_{\partial B_R(x_0)} = w_n|_{\partial B_R(x_0)}, \\ \Delta w_{2n}|_{\partial B_R(x_0)} = \Delta w_n|_{\partial B_R(x_0)}. \end{cases}$$

Note that  $w_{1n} > 0$ ,  $w_{2n} > 0$  in  $B_R(x_0)$  by the maximum principle.

First, we will show that  $w_{2n}$  is uniformly bounded near  $x_0$ . Indeed, by the mean value theorem for biharmonic function, we know

$$\|w_{2n}\|_{L^\infty(B_{R/2}(x_0))} \leq C\|w_n\|_{L^1(B_R(x_0))} \leq C\|w_n\|_{L^1(\Omega)} \leq C$$

holds true. Here the last inequality follows by Lemma 4.1.

So we need only to derive the boundedness of  $w_{2n}$ .

For this purpose, choose  $t > 1$  such as  $t' = t/(t-1) = L_0 + (\delta/2)$ . Note that the fact  $L_0 > 1$  permits the existence of such  $t > 1$ . Then we have

$$\int_{B_R(x_0)} f_n dx < \frac{16\sigma_4}{L_0 + \delta} < \frac{16\sigma_4}{t'}.$$

Brezis-Merle type  $L^1$ -estimate (Lemma 4.1) implies

$$(4.2) \quad \int_{B_R(x_0)} \exp(t'w_{1n}(x)) dx < C,$$

where  $C = C(\delta) \rightarrow \infty$  as  $\delta \rightarrow 0$ .

By an elementary inequality  $\log x \leq x/e$  for  $x > 0$ , we have

$$\begin{aligned} \log f_n &= \log \frac{u_n^{p_n}}{\lambda_n} = p_n \log \frac{u_n}{\lambda_n^{1/p_n}} \leq p_n \frac{u_n}{e\lambda_n^{1/p_n}} \\ &\leq \frac{L_0 + (\delta/3)}{\lambda_n} \frac{u_n}{\lambda_n^{1/p_n}} = \frac{t' - (\delta/6)}{\lambda_n^{1/p_n}} \frac{u_n}{\lambda_n} \leq t'w_n(x), \end{aligned}$$

here, the second inequality follows by definition of  $L_0$ :

$$\frac{p_n\lambda_n}{e} < L_0 + \frac{\delta}{3}$$

for  $n$  large, and the third inequality follows from the fact

$$\lim_{n \rightarrow \infty} \lambda_n^{1/p_n} = 1$$

which in turn follows from  $\lambda_n = O(1/p_n)$  (Corollary 3.1).

Thus, we get a pointwise estimate

$$f_n(x) < \exp(t'w_n(x))$$

which implies

$$(4.3) \quad (f_n e^{-w_{1n}})^t < C e^{t'w_{1n}}$$

on  $B_{R/2}(x_0)$ , because  $w_{2n}$  is bounded uniformly on  $B_{R/2}(x_0)$ .

Rewrite the equation satisfied by  $w_{1n}$  as

$$\begin{cases} \Delta^2 w_{1n} = \underbrace{f_n e^{-w_{1n}(x)}}_{V_n(x)} e^{w_{1n}} & \text{in } B_R(x_0), \\ w_{1n}|_{\partial B_R(x_0)} = \Delta w_{1n}|_{\partial B_R(x_0)} = 0. \end{cases}$$

Now, we check the assumptions of Corollary 4.2.

(1)  $V_n = f_n e^{-w_{1n}}$  is uniformly bounded in  $L^t(B_{R/2}(x_0))$  by (4.2) and (4.3).

(2)  $\|w_{1n}\|_{L^1(B_{R/2}(x_0))} \leq C$  by Lemma 4.1.

(3)  $\int_{B_{R/2}} V_n e^{w_{1n}} dx = \int_{B_{R/2}} f_n dx \leq \exists \varepsilon_0 < 16\sigma_4/t'$  by the definition of  $t$ .

Therefore by applying Corollary 4.2 to  $w_{1n}$  on  $B_{R/2}(x_0)$ , we conclude that  $\{w_{1n}\}$  is uniformly bounded in  $L^\infty(B_{R/4}(x_0))$ .  $\square$

**Lemma 4.4.**  $S = \Sigma(\delta)$  for any  $\delta > 0$ .

Proof.  $S \subset \Sigma(\delta)$  is clear from Lemma 4.3.

On the other hand, suppose  $x_0 \in \Sigma(\delta)$  and  $\|w_n\|_{L^\infty(B_{R_0}(x_0))} < C$  for some  $C$  independent of  $n$ . Then  $f_n = \lambda_n^{p_n-1} w_n^{p_n} \rightarrow 0$  uniformly on  $B_{R_0}(x_0)$ , which implies  $x_0$  is a  $\delta$ -regular point;  $x_0 \notin \Sigma(\delta)$ .

This contradiction shows that for every  $R > 0$ ,

$$\lim_{n \rightarrow \infty} \|w_n\|_{L^\infty(B_R(x_0))} = \infty$$

holds at least for a subsequence. So  $x_0 \in S$ .  $\square$

By this Lemma, we obtain

$$1 \geq \mu(\Omega) \geq \frac{16\sigma_4}{L_0 + 2\delta} \text{card}(\Sigma(\delta)) = \frac{16\sigma_4}{L_0 + 2\delta} \text{card}(S).$$

Combining with the estimate  $L_0 \leq 32\sigma_4$ , we have

$$\text{card}(S) \leq \frac{L_0 + 2\delta}{16\sigma_4} \leq \frac{32\sigma_4 + 2\delta}{16\sigma_4},$$

hence

$$1 \leq \text{card}(S) \leq 2.$$

From now on, we prove the latter half of Theorem 1.2, so assume  $\text{card}(S) = 1$  and  $S = \{x_0\}$ ,  $x_0 \in \Omega$ .

Then  $w_n(x) \leq C$  on any compact set  $K \subset \overline{\Omega} \setminus \{x_0\}$ , which implies  $f_n \rightarrow 0$  compact uniformly on  $\overline{\Omega} \setminus \{x_0\}$ .

Take  $\varphi \in C_0(\Omega)$ . For given  $\varepsilon > 0$ , first we choose  $r > 0$  so small and then  $n \rightarrow \infty$ , we have

$$\begin{aligned} \left| \int_{\Omega} f_n \varphi \, dx - \varphi(x_0) \right| &\leq \int_{\Omega} f_n |\varphi(x) - \varphi(x_0)| \, dx \\ &\leq \int_{B_r(x_0)} f_n |\varphi(x) - \varphi(x_0)| \, dx + \int_{\Omega \setminus B_r(x_0)} f_n |\varphi(x) - \varphi(x_0)| \, dx \leq \varepsilon. \end{aligned}$$

Therefore

$$(4.4) \quad f_n \xrightarrow{*} \delta_{x_0}$$

in the sense of Radon measures of  $\Omega$ . Thus Theorem 1.2 (1) is proved.

Set  $\tilde{w}_n := -\Delta w_n$ . Then

$$\begin{cases} -\Delta \tilde{w}_n = f_n & \text{in } \Omega, \\ \tilde{w}_n|_{\partial\Omega} = 0, \end{cases}$$

and  $f_n \rightarrow 0$  uniformly on any compact  $K \subset \overline{\Omega} \setminus \{x_0\}$ .

By elliptic regularity, there is a subsequence  $\{w_n\}, \{\tilde{w}_n\}$  (still denoted by the same symbols) and a function  $G$  such that

$$\begin{aligned} w_n &\rightarrow G \text{ in } C^{4,\alpha}(K), & \tilde{w}_n &\rightarrow -\Delta G \text{ in } C^{2,\alpha}(K), \\ w_n &\rightarrow G \text{ in } L^1(\Omega), & \tilde{w}_n &\rightarrow -\Delta G \text{ in } L^1(\Omega) \end{aligned}$$

since  $W^{1,q}(\Omega)$  boundedness of  $w_n, \tilde{w}_n$  ( $1 < q < 4/3$ ) and the compactness of the imbedding  $W^{1,q}(\Omega) \hookrightarrow L^1(\Omega)$ .

By integration by parts and (4.4), we see  $G = G_4(\cdot, x_0)$ . So Theorem 1.2 (2) is proved.

Finally, to prove the characterization of blow up point, we need the famous

**Lemma 4.5** (Pohozaev identity). *Let  $u$  be a  $C^4$ -solution of  $\Delta^2 u = f(u)$  in  $\Omega$ . Then we have for any  $y \in \mathbf{R}^4$ ,*

$$\begin{aligned} 4 \int_{\Omega} F(u) \, dx &= \int_{\partial\Omega} (x+y, v) F(u) \, d\sigma + \frac{1}{2} \int_{\partial\Omega} v^2 (x+y, v) \, d\sigma + 2 \int_{\partial\Omega} \frac{\partial u}{\partial v} v \, d\sigma \\ &\quad + \int_{\partial\Omega} \frac{\partial v}{\partial v} (x+y, \nabla u) + \frac{\partial u}{\partial v} (x+y, \nabla v) \, d\sigma \\ &\quad - \int_{\partial\Omega} (\nabla u, \nabla v) (x+y, v) \, d\sigma. \end{aligned}$$

In particular, we have

$$\int_{\partial\Omega} v F(u) \, d\sigma + \frac{1}{2} \int_{\partial\Omega} v^2 v \, d\sigma + \int_{\partial\Omega} \left\{ \frac{\partial v}{\partial v} \nabla u + \frac{\partial u}{\partial v} \nabla v - (\nabla u, \nabla v) v \right\} \, d\sigma = \vec{0}.$$

Here,  $F(u) = \int_0^u f(s) ds$ ,  $v = -\Delta u$  and  $\nu(x)$  is the unit outer normal vector at  $x \in \partial\Omega$ .

More general version of this formula can be seen, for example, in [7]. In our case, integrating the identity on  $\Omega$

$$\begin{aligned} & \operatorname{div}((x+y, \nabla v) \nabla u + (x+y, \nabla u) \nabla v - (\nabla u, \nabla v)(x+y)) \\ &= (x+y, \nabla v) \Delta u + (x+y, \nabla u) \Delta v - 2(\nabla u, \nabla v) \end{aligned}$$

for  $u, v \in C^2(\overline{\Omega})$ ,  $\nabla = \nabla_x$ , and noting that

$$\operatorname{div}((x+y)F(u)) = f(u)(x+y, \nabla u) + 4F(u)$$

and

$$\operatorname{div}\left(\frac{1}{2}v^2(x+y) + 2v\nabla u\right) = v(\nabla v, x+y) + 2(\nabla u, \nabla v)$$

if  $v = -\Delta u$ , we get the desired formula.

Take  $r > 0$  so small that  $B_r(x_0) \subset \Omega$ . First, apply the Pohozaev identity to  $w_n$  on  $\Omega$ . Note that  $w_n$  solves  $\Delta^2 w_n = f_n = \lambda_n^{p_n-1} w_n^{p_n}$  in  $\Omega$ . So  $F(w_n) = (\lambda_n^{p_n-1}/(p_n+1))w_n^{p_n+1}$  in the following identity.

$$\begin{aligned} & \int_{\partial\Omega} v F(w_n) d\sigma + \frac{1}{2} \int_{\partial\Omega} (\Delta w_n)^2 v d\sigma + \int_{\partial\Omega} \left\{ \frac{\partial(-\Delta w_n)}{\partial v} \nabla w_n + \frac{\partial w_n}{\partial v} \nabla(-\Delta w_n) \right\} d\sigma \\ & - \int_{\partial\Omega} \{(\nabla w_n, \nabla(-\Delta w_n))v\} d\sigma \\ & = \vec{0}. \end{aligned}$$

Since  $w_n = \Delta w_n = 0$  on  $\partial\Omega$ , we have

$$\int_{\partial\Omega} \left\{ \frac{\partial(-\Delta w_n)}{\partial v} \nabla w_n + \frac{\partial w_n}{\partial v} \nabla(-\Delta w_n) - (\nabla w_n, \nabla(-\Delta w_n))v \right\} d\sigma = \vec{0}.$$

Now we know that

$$w_n \rightarrow G_4(\cdot, x_0) \quad \text{in } C_{loc}^4(\overline{\Omega} \setminus \{x_0\}), \quad -\Delta w_n \rightarrow G_2(\cdot, x_0) \quad \text{in } C_{loc}^2(\overline{\Omega} \setminus \{x_0\})$$

as  $n \rightarrow \infty$ . Therefore letting  $n \rightarrow \infty$ , we have:

$$(4.5) \quad \int_{\partial\Omega} \left\{ \frac{\partial G_2}{\partial v} \nabla G_4 + \frac{\partial G_4}{\partial v} \nabla G_2 - (\nabla G_4, \nabla G_2)v \right\} d\sigma = \vec{0}.$$

On the other hand, apply the Pohozaev identity to  $G_4(\cdot, x_0)$  on  $\Omega \setminus B_r(x_0)$ . Note that  $G_4(\cdot, x_0)$  solves  $\Delta^2 G_4 = 0$  in  $\Omega \setminus B_r(x_0)$  and  $G_4 = G_2 = 0$  on  $\partial\Omega$ . Therefore we



have

$$\frac{1}{2} \int_{\partial(\Omega \setminus B_r(x_0))} G_2^2 v \, d\sigma + \int_{\partial(\Omega \setminus B_r(x_0))} \left\{ \frac{\partial G_2}{\partial \nu} \nabla G_4 + \frac{\partial G_4}{\partial \nu} \nabla G_2 - (\nabla G_4, \nabla G_2) v \right\} d\sigma = \vec{0},$$

that is,

$$(4.6) \quad \begin{aligned} & \frac{1}{2} \int_{\partial B_r(x_0)} G_2^2 v \, d\sigma + \int_{\partial B_r(x_0)} \left\{ \frac{\partial G_2}{\partial \nu} \nabla G_4 + \frac{\partial G_4}{\partial \nu} \nabla G_2 - (\nabla G_4, \nabla G_2) v \right\} d\sigma \\ &= \int_{\partial \Omega} \left\{ \frac{\partial G_2}{\partial \nu} \nabla G_4 + \frac{\partial G_4}{\partial \nu} \nabla G_2 - (\nabla G_4, \nabla G_2) v \right\} d\sigma. \end{aligned}$$

From (4.5) and (4.6), we have:

$$(4.7) \quad \begin{aligned} \vec{0} = & \underbrace{\frac{1}{2} \int_{\partial B_r(x_0)} G_2^2 v \, d\sigma}_{[A]} + \underbrace{\int_{\partial B_r(x_0)} \frac{\partial G_2}{\partial \nu} \nabla G_4 \, d\sigma}_{[B]} \\ & + \underbrace{\int_{\partial B_r(x_0)} \frac{\partial G_4}{\partial \nu} \nabla G_2 \, d\sigma}_{[C]} - \underbrace{\int_{\partial B_r(x_0)} (\nabla G_4, \nabla G_2) v \, d\sigma}_{[D]}. \end{aligned}$$

Substituting

$$\begin{aligned} G_4(x, x_0) &= H_4(x, x_0) - \frac{1}{4\sigma_4} \log |x - x_0|, \\ G_2(x, x_0) &= -\Delta G_4(x, x_0) = H_2(x, x_0) + \frac{1}{2\sigma_4} \frac{1}{|x - x_0|^2} \end{aligned}$$

in the above identity (4.7), we compute:

• **Estimate of [A]**

$$\int_{\partial B_r(x_0)} G_2^2 v \, d\sigma = \int_{\partial B_r(x_0)} \left( \frac{1}{2\sigma_4 |x - x_0|^2} + O(1) \right)^2 v \, d\sigma = O(r).$$

Note that  $v = (x - x_0)/|x - x_0|$  on  $\partial B_r(x_0)$  and

$$\int_{\partial B_r(x_0)} h(r) v \, d\sigma = \vec{0}$$

for any function  $h(r)$ ,  $r = |x - x_0|$ .

• **Estimate of [B]**

$$\int_{\partial B_r(x_0)} \frac{\partial G_2}{\partial \nu} \nabla G_4 \, d\sigma$$

$$\begin{aligned}
&= \int_{\partial B_r(x_0)} \left( \nabla H_2 \cdot \nu - \frac{1}{\sigma_4 |x - x_0|^3} \right) \left( \nabla H_4 - \frac{1}{4\sigma_4} \frac{x - x_0}{|x - x_0|^2} \right) d\sigma \\
&= -\frac{1}{\sigma_4 r^3} \int_{\partial B_r(x_0)} \nabla H_4(x, x_0) d\sigma + O(r^2) \\
&= -\nabla H_4(x^*, x_0) + O(r^2)
\end{aligned}$$

where  $\exists x^* \in \partial B_r(x_0)$ . The last equality comes from the mean value theorem of integral.

• **Estimate of [C]**

$$\begin{aligned}
&\int_{\partial B_r(x_0)} \frac{\partial G_4}{\partial \nu} \nabla G_2 d\sigma \\
&= \int_{\partial B_r(x_0)} \left( \nabla H_4 \cdot \nu - \frac{1}{4\sigma_4 |x - x_0|} \right) \left( \nabla H_2 - \frac{1}{\sigma_4} \frac{x - x_0}{|x - x_0|^4} \right) d\sigma \\
&= -\frac{1}{\sigma_4 r^3} \int_{\partial B_r(x_0)} \nu (\nabla H_4 \cdot \nu) d\sigma + O(r^2).
\end{aligned}$$

• **Estimate of [D]**

$$\begin{aligned}
&\int_{\partial B_r(x_0)} (\nabla G_4, \nabla G_2) \nu d\sigma \\
&= \int_{\partial B_r(x_0)} \left( \nabla H_4 - \frac{1}{4\sigma_4} \frac{x - x_0}{r^2}, \nabla H_2 - \frac{1}{\sigma_4} \frac{x - x_0}{r^4} \right) \nu d\sigma \\
&= -\frac{1}{\sigma_4 r^3} \int_{\partial B_r(x_0)} \nu (\nabla H_4 \cdot \nu) d\sigma + O(r^2).
\end{aligned}$$

Letting  $r \rightarrow 0$  and noting that  $x^* \rightarrow x_0$ , we have

$$\vec{0} = -\nabla H_4(x_0, x_0).$$

This proves Theorem 1.2 (3) □

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NOTE ADDED IN THE PROOF. Recently, we have succeeded to confirm the conjecture in Remark 1.3 (2) affirmatively. Please refer to “Single-point condensation phenomena for a four-dimensional biharmonic Ren-Wei problem” (preprint).

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